Tutorial 11

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1. **Dirichlet Principle**
$$\mathcal{A} = \{w \in C^1(\bar{D}) \cap C^2(D) : w = f(x) \text{ on } \partial D\}$$
The Dirichlet Energy
$$E(w) = \frac{1}{2} \iiint_D |\nabla u|^2 dx$$
If $u(x) \in \mathcal{A}$ satisfies $E(u) \leq E(w) \quad \forall w \in \mathcal{A}$. Show that $\Delta u = 0$ in D.

Proof: Let $v(x)$ be any smooth function that vanishes on ∂D. Then $u(x) + \epsilon v(x) \in \mathcal{A}$,
$$E(u + \epsilon v) = \frac{1}{2} \iiint_D |\nabla(u + \epsilon v)|^2 dx = E(u) + \epsilon \iiint_D \nabla u \cdot \nabla v + \epsilon^2 E(v)$$
By using the first Green's Formula and $v = 0$ on ∂D, we have
$$E(u + \epsilon v) = E(u) - \epsilon \iiint_D \Delta uv + \epsilon^2 E(v)$$
Since the minimum occurs for $\epsilon = 0$, hence $\frac{d}{d\epsilon} E(u+\epsilon v)\Big|_{\epsilon=0} = -\iiint_D \Delta uv = 0$ for any smooth function $v(x)$. Let $v(x) \equiv 1$ on D' where $\bar{D}' \subset D$ and $v(x) \equiv 0$ on D/D'. Suppose $v_n(x)$ is a family of smooth functions that converges to $v(x)$ almost everywhere on D as $n \to \infty$. Thus
$$0 = \lim_{n \to \infty} \iiint_D \Delta uv_n dx = \iiint_D \lim_{n \to \infty} \Delta uv_n = \iiint_D \Delta uv = \iiint_{D'} \Delta u$$
(where we have used the Lebesgue Dominated Covergence Theorem.) Hence by Second Vanishing Theorem, $\Delta u = 0$ in D.

2. Solve $u_{xx} + u_{yy} = 0$ in the wedge $r < a,\ 0 < \theta < \beta$ with the BCs
$$u = \theta \quad \text{on } r = a, \qquad u = 0 \quad \text{on } \theta = 0, \qquad \text{and } u = \beta \quad \text{on } \theta = \beta.$$
(*Hint:* Look for a function independent of r.)

Solution: It is obvious that $u(r, \theta) = \theta$ is a solution. Hence, by the uniqueness theorem, $u(r, \theta) = \theta$ is the unique solution.